MODELING THE LORENZ EQUATION ON AN ELECTRICAL CIRCUIT

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1. Introduction

Formally, chaotic systems must exhibit three properties:

(1) sensitivity to the initial conditions,
(2) dense periodic orbits, and
(3) topological transitivity.

What this means, generally, is that the resulting states of a chaotic system change greatly under small perturbations.

One such chaotic system is the Lorenz system,

$$\begin{aligned} \dot{x} &= sy - sx \\ \dot{y} &= rx - xz - y \\ \dot{z} &= xy - bz, \end{aligned} \tag{1.1}$$

formulated by Edward Lorenz in 1963 as a simplified model of atmospheric convection. Upon running a punch card with rounded off initial conditions through an old computer to evaluate this equation, he noticed the outcome of the simulation to be completely different than when begun with the fully written out conditions. It was this accidental observation that he noticed the system to be chaotic.

Using this canonical example of chaos, we will explore here a methodology of reproducing chaotic systems within circuitry. While more complicated circuits could be constructed, the simplest ways to achieve this goal is through operational amplifiers (op amps), and voltage multipliers. Three op amp integrators can be used for integration of each component (x, y, z) of Equation 1.1, and two voltage multipliers can be used for every non linear term. Resistors can then be used to scale terms. The resulting circuit can be seen in Figure 1.

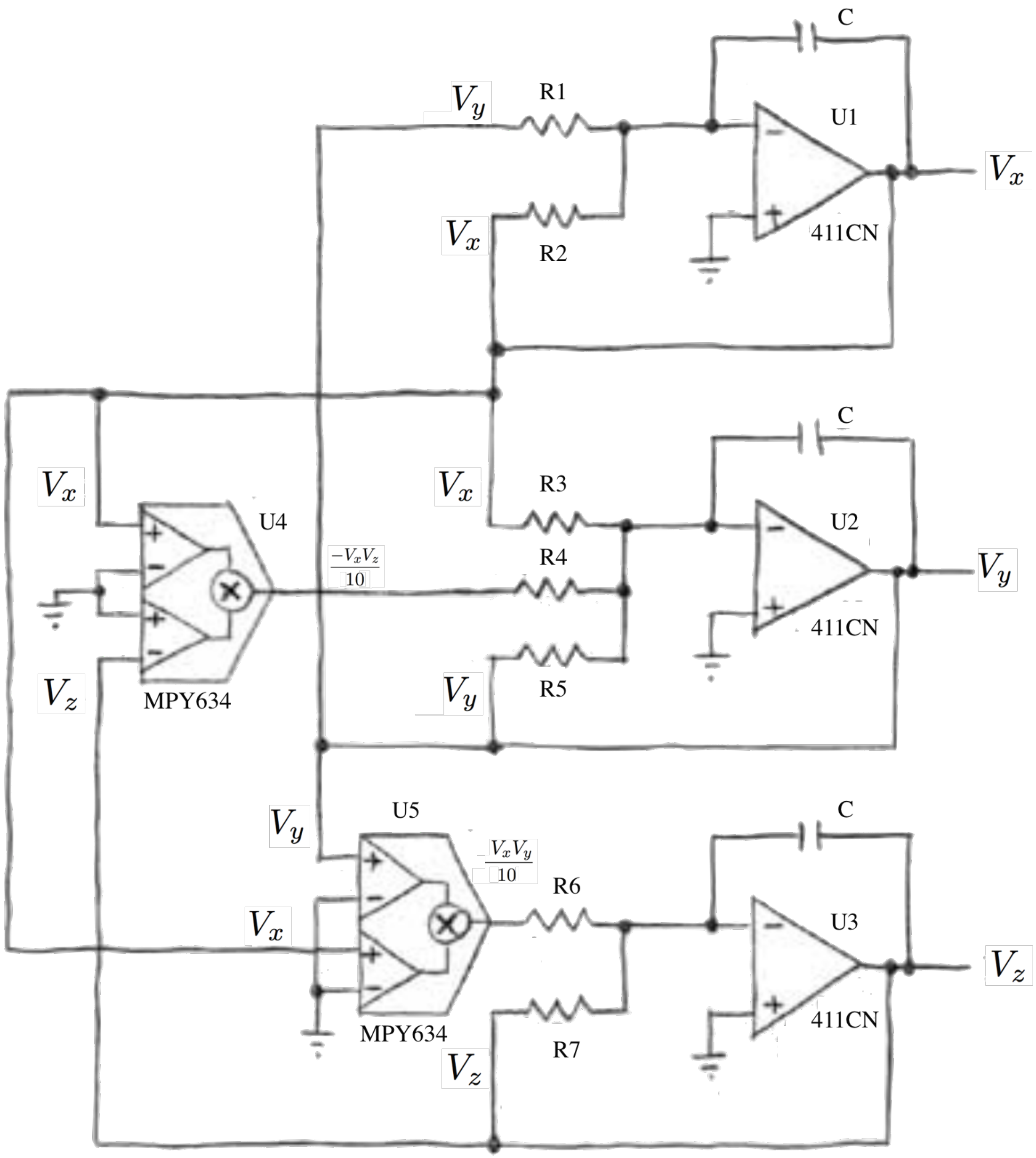

FIGURE 1. Circuit diagram of the lorenz circuit. Three op amps are used as integrators of V_x, V_y and V_z and two multipliers are used for the nonlinear terms in the circuit. This representation of the Lorenz system negates y. That is, the output V_y must be negated again upon measurement to obtain the true value y.

Within this circuit, voltages represent the quantities $x, -y$, and z, as integrating over $-y$ simplifies the circuit. As unitless quantities, x, y, and z are related to the voltages seen in the circuit, V_x, V_y, and V_z through some scaler factor $a\,\mathrm{V}$, and there values are integrated over time using the three op amp integrators. To understand how this works, consider the integrative circuit in Figure 1. Within this circuit, $V_1 = I_1R_1$, $V_2 = I_2R_2$, $V_3 = I_3R_3$, and $V_{\text{out}} = Q/C$. Given that $I_4 = I_1 + I_2 + I_3$, as no current is drawn by the input of the op amp, these equations can be combined, and generalized, to form

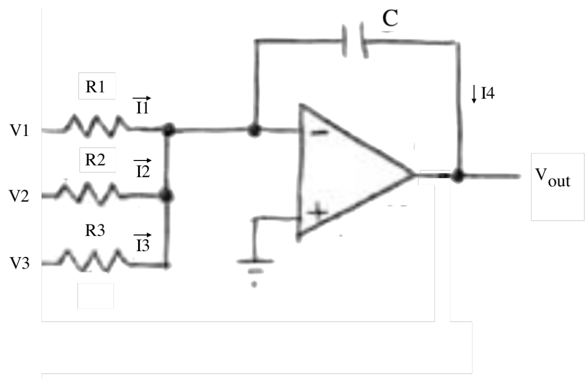

FIGURE 2. A sample integrator circuit with multiple inputs. Circuits such as this can be used to model a differential equation with multiple terms, following Equation 1.2.

$$V_{\text{out}} = -\int_0^t \frac{1}{C}\sum_{i=0}^{3}\frac{V_i}{R_i}dt, \tag{1.2}$$

for i voltage inputs. With a change of variables, $t = Ct'$, the factor of C can be neglected. This simple but powerful equation allows us to reconstruct the terms we see in Equation 1.1 with values that appear in the actual circuit. With a look at the circuit, taking particular care to keep track of all of the minus signs, it follows from Equation 1.2 that

$$\begin{aligned}
\frac{d}{dt'}V_x &= -\frac{V_y}{R_1} - \frac{V_x}{R_2}\\
\frac{d}{dt'}V_y &= -\frac{V_x}{R_3} + \frac{V_xV_z}{10R_4} - \frac{V_y}{R_5}\\
\frac{d}{dt'}V_z &= -\frac{V_xV_y}{10R_6} - \frac{V_z}{R_7},
\end{aligned}$$

where the factor of 10 in each of the multiplied terms comes from the MPY634, which outputs $\frac{AB}{10}$ for inputs A and B. Incorporating the scaling factors $Ct = t'$ and $V_{\{x,y,z\}} = a\{x, -y, z\}$, this becomes,

$$\begin{aligned} \frac{d}{dt'}x &= \left(\frac{y}{R_1} - \frac{x}{R_2}\right) \\ \frac{d}{dt'}y &= \left(\frac{x}{R_3} - \frac{axz}{10R_4} - \frac{y}{R_5}\right) \\ \frac{d}{dt'}z &= \left(\frac{axy}{10R_6} - \frac{z}{R_7}\right), \end{aligned} \tag{1.3}$$

which gives $\vec{R} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s} & \frac{1}{r} & \frac{a}{10} & 1 & \frac{a}{10} & \frac{1}{b} \end{bmatrix}$ when compared to Equation 1.1. Now, any scaler multiple of $\vec{R}$ will represent Equation 1.1, however, it will necessitate another rescaling of time. Assume we take all resistor values to be scaled by R_{ref}, Equation 1.3 becomes

$$\begin{aligned} \frac{d}{dt''}x &= sy - sx \\ \frac{d}{dt''}y &= rx - xz - y \\ \frac{d}{dt''}z &= xy - bz, \end{aligned} \tag{1.4}$$

for $t'' = CR_{\text{ref}}\, t$.

Notice that for any fixed point solution of this equation, (x_f, y_f, z_f), such that $\frac{d}{dt''}\{x, y, z\} = 0$,

$$\begin{aligned} x_f &= y_f \\ r &= z_f + 1 \\ b &= \frac{x_f y_f}{z_f}. \end{aligned} \tag{1.5}$$

No inference can be made about s, but it follows from these equations that $b(r-1) = x_f^2$ for any fixed point. No physical solution exists for $r < 1$ other than the trivial solution. However, for $r > 1$, fixed points exist at $x_f = y_f = \pm\sqrt{b(r-1)}$, $z_f = b - 1$.

2. Setup

A series of regimes for the Lorenz system were measured, including the fixed point regime for $r < 1$, a multiple fixed point regime for $r > 1$, the canonical chaotic attractor regime for $s = 10$, $r = 28$, and $b = 8/3$, and the periodic regime in the neighborhood of $s = 10$, $r = 85.25$, and $b = 8/3$. Across all regimes, $\vec{R} = \begin{bmatrix} 100 & 100 & R_3 & 10 & 1000 & 10 & 370 \end{bmatrix}$ kΩ was used, such that $s = 10$ and $b = 2.70$, with R_3 being used to vary r, thereby bringing the system through each regime. Per the scaling of time, we used $C = 0.37\,\mu$F and $R_{\text{ref}} = 1\,$MΩ for each measurement, for a total scaling of $t'' = .37\, t$.

For measurement of this circuit on an Arduino, we choose $a = 0.1$, to bring voltages within the neighborhood of $\pm 5\,$V. Such scaling is still insufficient,

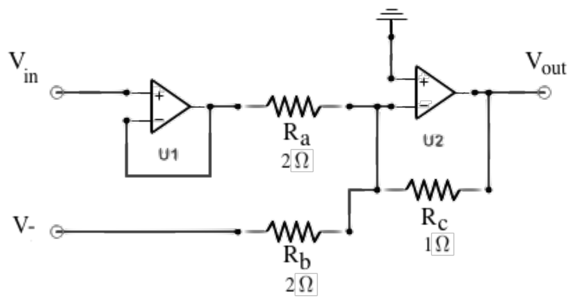

FIGURE 3. Above is the diagram for the mapping circuit. $U1$ acts as a follower, to segment the mapping circuit from the Lorenz circuit, while $U2$ carries out the mapping $V_{\text{out}} = -R_c\left(\frac{V_{\text{in}}}{R_a} - \frac{5}{R_b}\right)$.

as an Arduino can only read input voltages between $0-5\,\text{V}$. In order to map the incoming voltages into a range visible by the Arduino, we designed a circuit that used three resistors to map $F : V \to -R_c\left(\frac{V}{R_a} + \frac{V_-}{R_b}\right)$, for some translative voltage V_-, which is shown below in Figure 1. This mapping circuit was used on each component output of the circuit, with V_-, R_a, R_b and R_c being changed for each regime accordingly.

In Table 1, the parameters used to measure each regime are listed. [1]

TABLE 1. A collection of the parameters used for each regime measured. This includes the fixed point (FP) for $r < 1$, the fixed points for $r > 1$, the attractor, and a periodic state.

	FP r<1	FP r>1	Periodic	Attractor
R1 (kΩ)	100	100	100	100
R2 (kΩ)	100	100	100	100
R3 (kΩ)	2000	10.03	11.72	36
R4 (kΩ)	10	10	10	10
R5 (kΩ)	1000	1000	1000	1000
R6 (kΩ)	10	10	10	10
R7 (kΩ)	370	370	370	370
Ra (kΩ)	2	2	86	2
Rb (kΩ)	2	2	86	2
Rc (kΩ)	1	1	1	1
V- (V)	-5	- 5	- 7.78	- 5

[1]For the periodic regime, an $860\,\Omega$ resistor was used mistakenly for R_b in the z mapping circuit, in place of an $86\,\text{k}\Omega$ resistor.

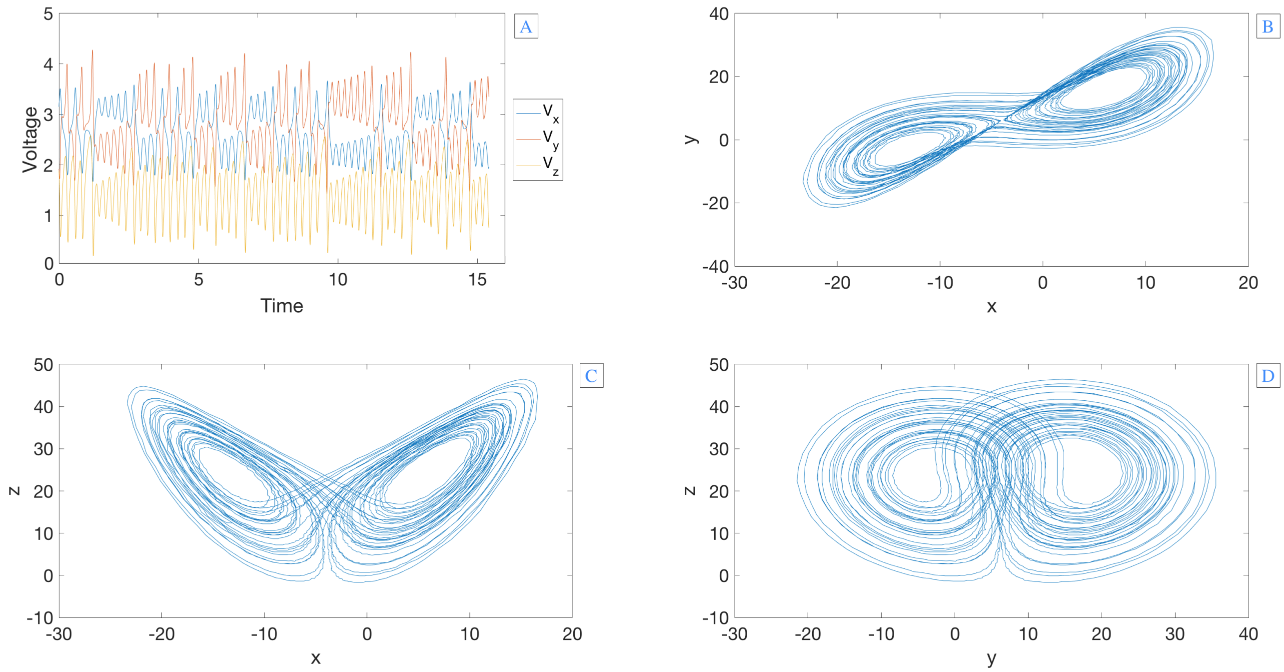

FIGURE 4. The measured output for $R_3 = 36\,\mathrm{k}\Omega$, $R_c = 1\,\mathrm{k}\Omega$, $R_a = R_b = 2\,\mathrm{k}\Omega$, and $V_{-} = -5\,\mathrm{V}$. As expected, the attractor displays the iconic owl mask shape when projected on the xz plane.

3. Results

In the attractor regime, the output voltages $V_{\{x,y,z\}}$ were measured over the course of sixteen seconds, from non-zero initial conditions (the system was let run before taking data). The results of this measurement are shown in Figure 2. The raw data was transformed from (V_x, V_y, V_z, t'') to (x, y, z, t) using $t\frac{1}{CR_{\text{ref}}}t = \frac{1}{.37}t$ and $\{x, -y, z\} = \frac{1}{a}F^{-1}V_{\{x,y,z\}} = 50 - 20V_{\{x,y,z\}}$. The plots for these unit less variables are shown in Plots $B - C$ of Figure 2.

Further measurements were taken regarding fixed points of the Lorenz system in both the $r < 1$ and $r > 1$ regimes. First we measured the fixed point for $r < 1$, using $r = .5$ to collapse the system into what should be the trivial solution $\vec{x}_f = \vec{0}$, however, this equilibrium position was not observed. Instead the system remained stationary at the point $\vec{x_f} = \begin{bmatrix} 2.5 & 10.3 & 3.8 \end{bmatrix}$, plotted in figure 3.

The behavior of the multiple fixed point regime in the $r > 1$ range was also irregular. Two fixed points were observed for $R_3 = 11.73\,\mathrm{k}\Omega$, as expected, but of unpredicted values. Equation 1.5 predicts fixed points at $\vec{x_f} = \begin{bmatrix} 15.07 & 15.07 & 1.7 \end{bmatrix}$ and $\vec{x_f} = \begin{bmatrix} -15.07 & -15.07 & 1.7 \end{bmatrix}$, while

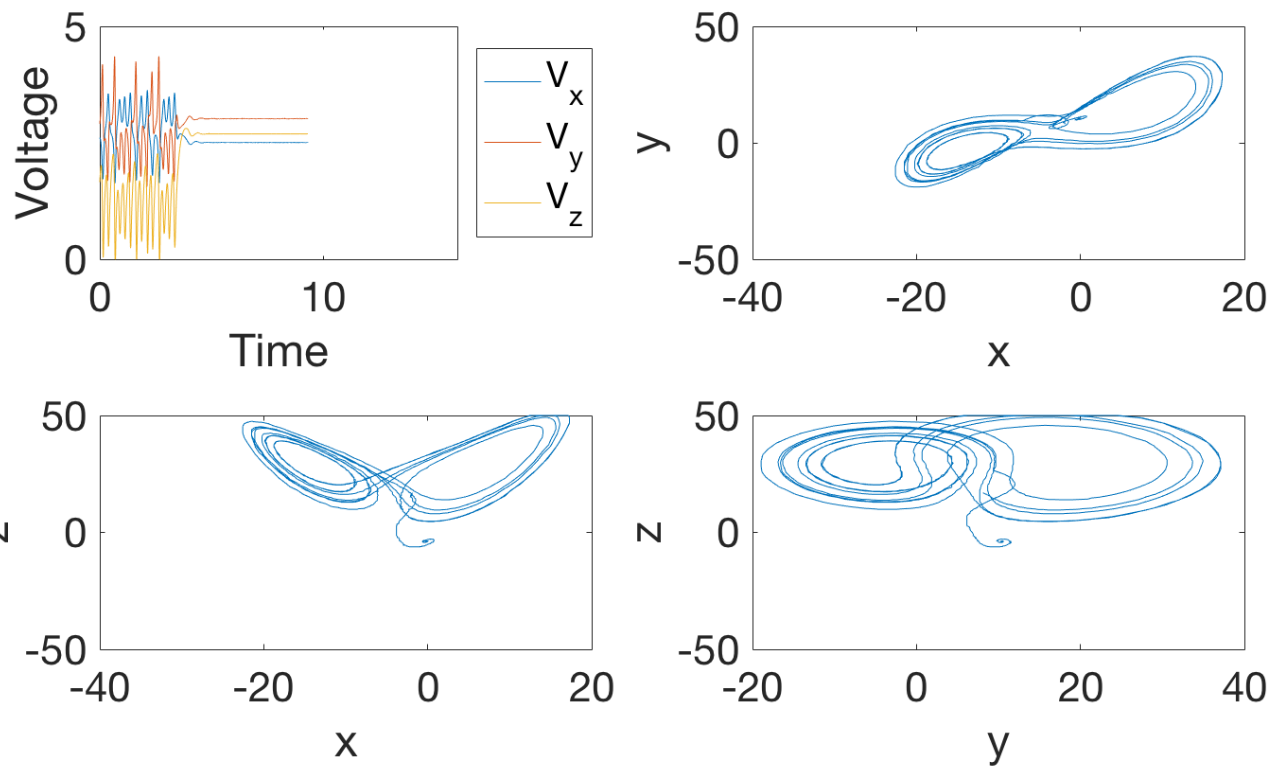

FIGURE 5. The measured output for $R_3 = 2\,\mathrm{M}\Omega$, $R_c = 1\,\mathrm{k}\Omega$, $R_a = R_b = 2\,\mathrm{k}\Omega$, and $V_- = -5\,\mathrm{V}$. A fixed point was observed close to the origin, however, this fixed point was offset from the origin on the x and z axes.

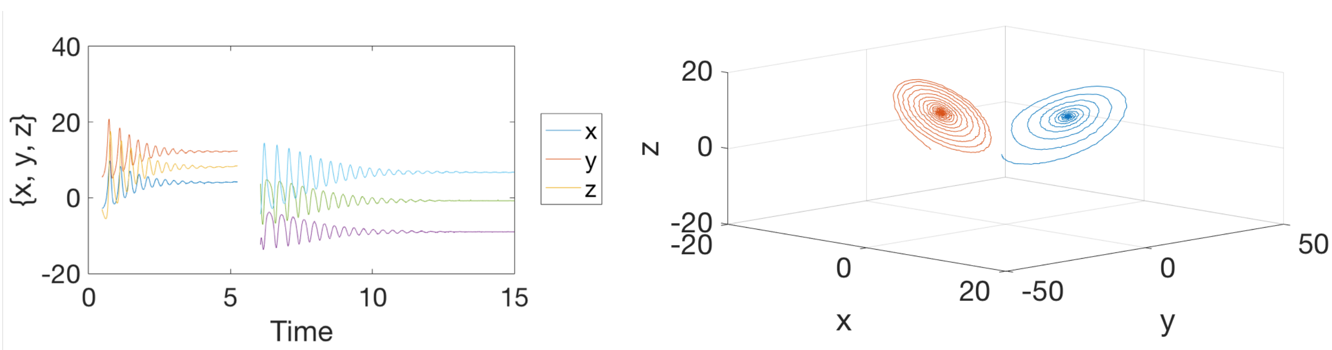

FIGURE 6. The measured output for $R_3 = 10.03\,\mathrm{k}\Omega$, $R_c = 1\,\mathrm{k}\Omega$, $R_a = R_b = 2\,\mathrm{k}\Omega$, and $V_- = -5\,\mathrm{V}$. Two fixed points were measured, however, they were not symmetric about $y = -x$ as predicted by Equation 1.5.

$\vec{x_f} = \begin{bmatrix} 4.2969 & 12.4023 & 8.4961 \end{bmatrix}$ and $\vec{x_f} = \begin{bmatrix} -8.9844 & -0.7812 & 6.6406 \end{bmatrix}$ were measured.

In the periodic regime, the output voltages were measured over the course of three seconds, again from non zero-initial conditions, the results of which are shown in Figure 3. The raw data was similarly mapped here, but due to an incorrectly chosen resistor, the mapping circuit did not work as expected, saturating one of the op amps and creating a visible ceiling on the z-axis.

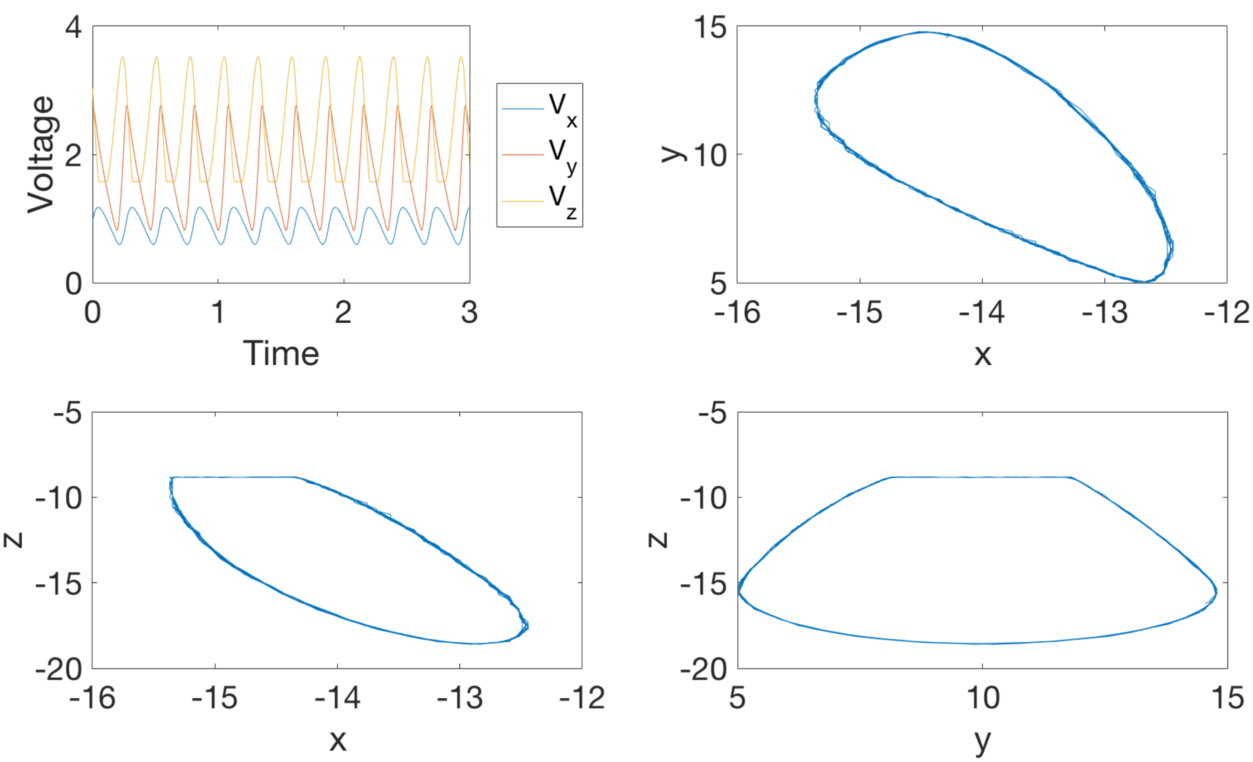

FIGURE 7. The measured output for $R_3 = 11.72\,\mathrm{k\Omega}$, $R_c = 1\,\mathrm{k\Omega}$, $R_a = R_b = 86\,\mathrm{k\Omega}$, and $V_- = -7.78\,\mathrm{V}$. At these values, the system follows a stable periodic orbit. There is apparent clipping on the z-axis, which occurred due to a $860\,\Omega$ resistor being used mistakingly on the z-axis mapping circuit, which saturated one of the op-amps.

4. Conclusion

It is clear from the attractor regime that this circuit was an effective integrator of the Lorenz system. The owl mask shape of the system was observed, even under perturbations of r. In this sense, we did construct a working representation of the Lorenz system. However, fixed point analysis proves that there are faults nonetheless with the construction of the circuit, or at least the interpretation of its outputs, as the measured values deviate from what was derived, even for the trivial solution. We believe these inaccuracies to stem, in part, from in the mapping circuit. Excluding human error in incorrectly inverting the effect of the mapping circuit, tolerances of resistors is one possible source of skew in the results. Even gold banded resistors can deviate the expected scaling of the circuit up to 10%. This does not explain the irregular results completely, but this could explain a portion of the skewing away from expected values. Inaccuracies must also be due to other, internal sources of error in the circuit, as the trivial solution was not found from the fixed point $r < 1$ regime even when measured prior to the mapping circuits.

Still, the measurements made here provide valuable insights into the methodology of integration using circuits. Other such systems can also be integrated in a similar manner, using Equation 1.2 for each component in need of integration. While integrative circuits are not needed in today's day and age for aiding in analysis, as software tools produce cleaner results much faster, with much better control over the initial conditions, integrative circuits have a variety of direct uses in electronics, such as in digital to analog converters and wave-shaping circuits. In cases such at these, where integration is better off-loaded to an integrated circuit, the practices formulated here prove to be valuable.